

SOLUTIONS OF STROMINGER SYSTEM FROM UNITARY REPRESENTATIONS OF COCOMPACT LATTICES OF $SL(2, \mathbb{C})$

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ABSTRACT. Given an irreducible unitary representation of a cocompact lattice of $SL(2, \mathbb{C})$, we explicitly write down a solution of the Strominger system of equations. These solutions satisfy the equation of motion, and the underlying holomorphic vector bundles are stable.

1. INTRODUCTION

Evoking physical requirements from anomaly cancellations, realistic fermionic spectrum and the appropriate amount ($N = 1$) of Space-time supersymmetry, Candelas *et al* had originally proposed a model for compactification of the superstring, by analyzing the vacuum configurations of these 10-dimensional theories [CHSW]. Anomaly cancellation requirements (which constrain the gauge groups of these models to be $O(32)$ or $E_8 \times E_8$), along with the requirement of a zero cosmological constant, then lead them to propose/construct the 10-dimensional vacuum solutions of these theories to be of the metric product type $X_4 \times \mathcal{M}$, where X_4 is the maximally symmetric $4d$ space-time (which should admit unbroken $N = 1$ supersymmetry), and \mathcal{M} is a complex 3-dimensional Calabi-Yau manifold. Subsequently, these conclusions were further generalized to include other gauge groups (like $SU(4)$ or $SU(5)$), as would arise when considering compactifications for the strongly coupled heterotic string theory. The correspondence between the algebro-geometric notion of stable vector bundles and the existence of Hermitian-Yang-Mills connections was one of the primary mathematical input underlying these derivations [Wi]. In all these examples, the supersymmetric vacuum (manifold) was assumed to be one whose geometry had no torsion. Hence the existence of a solution on such a given manifold was mostly a topological question and the issue of existence of appropriate solutions (obeying all the physical requirements) often boiled down to a set of conditions on the Chern classes of the vacuum manifold \mathcal{M} and the Yang-Mills Gauge connections.

In 1986, Strominger investigated the necessary and sufficient conditions for space-time supersymmetric solutions of the heterotic string. While considering more general space-times as solutions to the heterotic superstring solutions, Strominger, [St], was lead to considering vacuum configurations *with* torsion. He relaxed the requirement of the

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10-dimensional vacuum metric by considering that, for more general vacuum configurations (which can sustain non-zero fluxes as well as space-time supersymmetry), the 10-dimensional space-time be a *warped* product of X_4 and the 6-dimensional internal space \mathcal{M} . Analyzing the constraints imposed by the requirements of $N = 1$ space-time (*i.e.*, 4 dimensional) supersymmetry (and other usual consistency requirements like anomaly cancellation), Strominger then established that the 6-dimensional internal manifold \mathcal{M} should be a compact, connected, complex manifold (hereafter denoted as M), such that its canonical line bundle K_M is holomorphically trivial. Let $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ be a $(1, 1)$ Hermitian form on M , and let ∇^M be a connection on TM compatible with ω . We denote its curvature by R . Further, let E be a holomorphic vector bundle on M equipped with the (gauge) connection A , and corresponding curvature F_A . It turns out that the anomaly cancellation condition then demands that the Hermitian $(1, 1)$ form ω obeys an equation of the form:

$$\sqrt{-1} \partial \bar{\partial} \omega = \frac{\alpha'}{4} (\text{trace}(R \wedge R) - \text{trace}(F_A \wedge F_A)) .$$

The consistency conditions from requirements of the space-time supersymmetry translates into the equation:

$$d^* \omega = \sqrt{-1} (\bar{\partial} - \partial) \ln \|\Omega\|_\omega$$

for the Hermitian form ω and the holomorphic 3-form Ω . The previous equation may also be equivalently re-written as [LY2]:

$$d (\|\Omega\|_\omega \cdot \omega^2) = 0$$

The above equations, along with the system (constraining the Yang-Mills Gauge theory content):

$$F_A^{2,0} = F_A^{0,2} = 0, \quad F \wedge \omega^2 = 0$$

gives a complete and general solution of a superstring theory with torsion and with a flux that allows a non-trivial dilation field (cosmological constant). Henceforth, the above system of equations (which are derived solely from the explicit requirements stemming from Superstring theory) would be referred to as the *Strominger system* of equations. Thus, by considering vacuum geometries with torsion, Strominger was able to relax the requirement of M to be Kähler and consider more general complex 3-manifolds. But the price to be paid was that the familiar tools and methods from Kähler geometry could now no longer be applied to these more general cases. Moreover, a purely topological characterization and classification of these heterotic superstring vacua solutions (*i.e.*, the Chern classes of the bundles E and the vacuum manifold M), would no longer suffice.

The above results provide us with the necessary and sufficient conditions for any heterotic superstring theory solution (admitting space-time supersymmetry for its vacuum configuration) to exist, but in practice, it is quite a difficult matter to exhibit or actually explicitly construct a solution which exists (and satisfies the Strominger equation). Apart from its interest and usefulness in the context of string theory, it is also of interest from

a mathematical point of view to find solutions (*i.e.*, construct the bundles E with the appropriate connection A for a given manifold M with properties as defined above) of the Strominger system. In recent years, there has been a flurry of activities surrounding this problem of providing explicit constructive methods for solutions of these Strominger systems (cf. [AG1], [AG2], [Iv] and references therein). The present paper explores a new and altogether different constructive scheme, based on an approach that does not require the perturbative/deformation prescription.

Further attempts at exploring more general vacuum configurations for the heterotic string with non-zero fluxes have lead to some additional corrections to the original analysis of Strominger. These come from considering $(SU(3))$ instanton corrections at higher loops, and lead to the additional consistency conditions (for the solutions of the Strominger system) and these are:

$$R^{2,0} = R^{0,2} = 0, \quad R \wedge \omega^2 = 0.$$

These are referred to as equations of motion. Here we shall consider those solutions of the Strominger system which also additionally satisfy the above conditions.

In recent years, there has been a lot of activity, in trying to construct actual/explicit examples which are solutions to the above extended *Strominger system*. In [FTY], Fu, Tseng and Yau have studied the existence of smooth solutions to the Strominger system. They proposed a perturbation method where deformation theory results were used to construct solutions for some $U(4)$ and $U(5)$ principal bundles. Subsequent generalizations of this method lead to the construction of new examples (of solutions to the Strominger system) on a class of non-Kähler three-dimensional manifolds like T^2 -bundles over a $K3$ surface, or T^2 -bundles over Eguchi-Hanson spaces. Nevertheless finding new/more examples of such solutions has proved to be rather tricky, and it seems that there is no general ansatz/scheme for constructing an example; instead one has to invent specific prescriptions and construction procedure for every new example.

In the present work, we produce solutions of the Strominger system from irreducible unitary representations of any cocompact lattice in $SL(2, \mathbb{C})$. Let Γ be a cocompact lattice in $SL(2, \mathbb{C})$ (meaning $SL(2, \mathbb{C})/\Gamma$ is compact), and let $\rho : \Gamma \rightarrow U(n)$ be an irreducible homomorphism, meaning no nonzero proper linear subspace of \mathbb{C}^n is left invariant by the action of the image $\rho(\Gamma)$. The compact complex manifold $M := SL(2, \mathbb{C})/\Gamma$ has trivial canonical line bundle, and M is equipped with a natural Hermitian structure. The Chern connection on TM for this Hermitian structure has the following properties:

- (1) the torsion of the connection is totally skew-symmetric, meaning it is a section of $\bigwedge^3 TM$, and
- (2) the holonomy of the connection lies in $SU(3)$

(see Corollary 4.3). The homomorphism ρ produces a holomorphic vector bundle over M with a flat unitary connection. This vector bundle is stable; see Proposition 4.5. We

prove that all these together produce a solution of the Strominger system satisfying the equation of motion; the details are in Theorem 4.6.

2. STROMINGER SYSTEM OF EQUATIONS

We write down the Strominger system of equations in one place for the convenience of later reference in Section 4.

Let M be a compact connected complex manifold of dimension three such that the canonical line bundle $K_M := \bigwedge^3 \Omega_M^1$ is holomorphically trivial. Let

$$\Omega \in H^0(M, K_M)$$

be a nowhere vanishing holomorphic section. Let ω be a Hermitian $(1, 1)$ -form on M . Take a connection ∇^T on TM compatible with ω ; its curvature will be denoted by R . Let E be a holomorphic vector bundle on M equipped with a connection A . Let F_A be the curvature of A . Let d^* be the adjoint of d with respect to ω ; it sends smooth k forms on M to $k - 1$ forms.

The sextuple $(M, \Omega, \omega, \nabla^T, E, A)$ is said to solve the *Strominger system* if the following equations hold:

$$(2.1) \quad F_A^{2,0} = F_A^{0,2} = 0, \quad F \wedge \omega^2 = 0$$

$$(2.2) \quad d^* \omega = \sqrt{-1}(\bar{\partial} - \partial)\|\Omega\|_\omega$$

$$(2.3) \quad d(\|\Omega\|_\omega \cdot \omega^2) = 0$$

$$(2.4) \quad \sqrt{-1}\partial\bar{\partial}\omega = \alpha'(\text{trace}(R \wedge R) - \text{trace}(F_A \wedge F_A)), \quad \text{where } \alpha' \in \mathbb{C}.$$

A Strominger system $(M, \Omega, \omega, E, A)$ as above is said to solve the *equation of motion* if

$$(2.5) \quad R^{2,0} = 0 = R^{0,2} \quad \text{and} \quad R \wedge \omega^2 = 0.$$

3. INVARIANT FORMS ON $\text{SL}(2, \mathbb{C})$

Consider the complex Lie group $\text{SL}(2, \mathbb{C})$. Let h_0 be the Hermitian structure on the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of $\text{SL}(2, \mathbb{C})$ defined by

$$(3.1) \quad h_0(A, B) = \text{trace}(AB^*),$$

where $B^* = \overline{B}^t$. Note that the adjoint action of $\text{SU}(2)$ on $\mathfrak{sl}(2, \mathbb{C})$ preserves h_0 .

Using the right-translation invariant vector fields on $\text{SL}(2, \mathbb{C})$, we identify the holomorphic tangent bundle $T\text{SL}(2, \mathbb{C})$ with the trivial vector bundle

$$\text{SL}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \text{SL}(2, \mathbb{C})$$

with fiber $\mathfrak{sl}(2, \mathbb{C})$. Let h be the unique right-translation invariant Hermitian structure on $\text{SL}(2, \mathbb{C})$ such that

$$h|_{T_e \text{SL}(2, \mathbb{C})} = h_0,$$

where $e \in \mathrm{SL}(2, \mathbb{C})$ is the identity element. Let

$$(3.2) \quad \omega_h \in C^\infty(\mathrm{SL}(2, \mathbb{C}), \Omega_{\mathrm{SL}(2, \mathbb{C})}^{1,1})$$

be the Kähler form associated to the Hermitian structure h on $\mathrm{SL}(2, \mathbb{C})$. We note that $d\omega_h \neq 0$.

Proposition 3.1. *Let $\xi \in C^\infty(\mathrm{SL}(2, \mathbb{C}), \Omega_{\mathrm{SL}(2, \mathbb{C})}^{1,0} \oplus \Omega_{\mathrm{SL}(2, \mathbb{C})}^{0,1})$ be a complex 1-form on $\mathrm{SL}(2, \mathbb{C})$ such that*

- *the right-translation action of $\mathrm{SL}(2, \mathbb{C})$ on itself preserves ξ , and*
- *the left-translation action of $\mathrm{SU}(2)$ on $\mathrm{SL}(2, \mathbb{C})$ preserves ξ .*

Then

$$\xi = 0.$$

Proof. Since the holomorphic tangent space of $\mathrm{SL}(2, \mathbb{C})$ at $e \in \mathrm{SL}(2, \mathbb{C})$ is identified with $\mathfrak{sl}(2, \mathbb{C})$, the evaluation of ξ at e is an element of $\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C} = (\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})^*$; here we identify $(T_e^{0,1}\mathrm{SL}(2, \mathbb{C}))^*$ with $(T_e^{1,0}\mathrm{SL}(2, \mathbb{C}))^*$ by sending any u to its conjugate \bar{u} . Let

$$\xi_0 := \xi(e) \in \mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}$$

be the evaluation of ξ at e . The adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathfrak{sl}(2, \mathbb{C})$ produces an action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}$. In particular, we get an action of $\mathrm{SU}(2)$ on $\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}$. The two given conditions on ξ imply that this action of $\mathrm{SU}(2)$ on $\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}$ fixes the element ξ_0 .

Consider the nondegenerate symmetric bilinear pairing on $\mathfrak{sl}(2, \mathbb{C})$ defined by

$$(3.3) \quad (A, B) \longmapsto \mathrm{trace}(AB).$$

It produces an isomorphism of $\mathfrak{sl}(2, \mathbb{C})$ with $\mathfrak{sl}(2, \mathbb{C})^*$ that is equivariant for the actions of $\mathrm{SL}(2, \mathbb{C})$ on $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})^*$. Using this identification between $\mathfrak{sl}(2, \mathbb{C})^*$ and $\mathfrak{sl}(2, \mathbb{C})$, the above element ξ_0 gives an element

$$\tilde{\xi}_0 \in \mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}.$$

We note that $\tilde{\xi}_0$ is fixed by the adjoint action of $\mathrm{SU}(2)$, because

- ξ_0 is fixed by the action of $\mathrm{SU}(2)$ on $\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}$, and
- the isomorphism between $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})^*$ is $\mathrm{SL}(2, \mathbb{C})$ -equivariant.

But no nonzero element of $\mathfrak{sl}(2, \mathbb{C})$ is fixed by the adjoint action of $\mathrm{SU}(2)$ on $\mathfrak{sl}(2, \mathbb{C})$. This implies that there is no nonzero element of $\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ that is fixed by the action of $\mathrm{SU}(2)$, because $(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})^{\mathrm{SU}(2)} = \mathfrak{sl}(2, \mathbb{C})^{\mathrm{SU}(2)} \otimes_{\mathbb{R}} \mathbb{C}$. (For an $\mathrm{SU}(2)$ -module W , by $W^{\mathrm{SU}(2)}$ we denote the space of invariants for the action of $\mathrm{SU}(2)$ on W .) Hence we conclude that $\tilde{\xi}_0 = 0$. So, $\xi_0 = 0$. This implies that $\xi = 0$ because it is fixed by the right-translation action of $\mathrm{SL}(2, \mathbb{C})$ on itself. \square

Proposition 3.2. *Let ζ be a C^∞ complex 4-form on $\mathrm{SL}(2, \mathbb{C})$ such that*

- the right-translation action of $\mathrm{SL}(2, \mathbb{C})$ on itself preserves ζ , and
- the left-translation action of $\mathrm{SU}(2)$ on $\mathrm{SL}(2, \mathbb{C})$ preserves ζ .

Then there is constant $c \in \mathbb{C}$ such that

$$\zeta = c \cdot \omega_h \wedge \omega_h,$$

where ω_h is constructed in (3.2).

Proof. As in the proof of Proposition 3.1, the evaluation of ζ at e is an element

$$\zeta_0 \in \bigwedge^4(\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}).$$

The adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathfrak{sl}(2, \mathbb{C})$ produces an action of $\mathrm{SL}(2, \mathbb{C})$ on the complex line $\bigwedge^6(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$. Since $\mathrm{SL}(2, \mathbb{C})$ does not have any nontrivial character, this action of $\mathrm{SL}(2, \mathbb{C})$ on $\bigwedge^6(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$ is trivial. The adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ produces actions of $\mathrm{SL}(2, \mathbb{C})$ on $\bigwedge^4(\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C})$ and $\bigwedge^2(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$. Fixing a nonzero element of the line $\bigwedge^6(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$, we get an $\mathrm{SL}(2, \mathbb{C})$ -equivariant isomorphism of $\bigwedge^4(\mathfrak{sl}(2, \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C})$ with $\bigwedge^2(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$. Using this isomorphism, the above element ζ_0 gives an element

$$(3.4) \quad \widehat{\zeta}_0 \in \bigwedge^2(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}).$$

The two given conditions on ζ imply that the element $\widehat{\zeta}_0$ in (3.4) is fixed by the action of $\mathrm{SU}(2)$ on $\bigwedge^2(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$ (recall that $\mathrm{SL}(2, \mathbb{C})$ acts on $\bigwedge^2(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$).

Note that

$$\bigwedge^2(\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}) = (\bigwedge^2 \mathfrak{sl}(2, \mathbb{C}))^{\oplus 2} \oplus (\mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C}));$$

this decomposition is preserved by the action of $\mathrm{SL}(2, \mathbb{C})$. There is no nonzero element of $\bigwedge^2 \mathfrak{sl}(2, \mathbb{C})$ preserved by the action of $\mathrm{SU}(2)$. The subspace of $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$ defined by all elements fixed pointwise by the action of $\mathrm{SU}(2)$ is one-dimensional, and it is generated by the element of $\mathrm{Sym}^2(\mathfrak{sl}(2, \mathbb{C})) \subset \mathfrak{sl}(2, \mathbb{C})^{\otimes 2}$ given by the nondegenerate pairing in (3.3). This immediately implies that the space of smooth complex 4-forms on $\mathrm{SL}(2, \mathbb{C})$ satisfying the two conditions in the proposition is one dimensional.

Since the inner product h_0 on $\mathfrak{sl}(2, \mathbb{C})$ in (3.1) is $\mathrm{SU}(2)$ -invariant, it follows immediately that the Hermitian structure h on $\mathrm{SL}(2, \mathbb{C})$ is preserved by the left-translation action of $\mathrm{SU}(2)$ on $\mathrm{SL}(2, \mathbb{C})$. Hence the Kähler form ω_h on $\mathrm{SL}(2, \mathbb{C})$ is preserved by the left-translation action of $\mathrm{SU}(2)$ on $\mathrm{SL}(2, \mathbb{C})$. Recall that ω_h is also preserved by the right-translation action of $\mathrm{SL}(2, \mathbb{C})$ on itself. Therefore, $\omega_h \wedge \omega_h$ is a nonzero complex 4-form satisfying the two conditions in the proposition. Since the space of smooth complex 4-forms on $\mathrm{SL}(2, \mathbb{C})$ satisfying the two conditions in the proposition is one dimensional, we now conclude that ζ is a constant scalar multiple of $\omega_h \wedge \omega_h$. \square

Lemma 3.3. *The differential form ω_h in (3.2) satisfies the identity*

$$d(\omega_h \wedge \omega_h) = 0.$$

Proof. Using the identification between $T_e SL(2, \mathbb{C})$ and $sl(2, \mathbb{C})$, the evaluation of the 5-form $d(\omega_h^2)$ at e is an element of $\bigwedge^5(sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})^*$; as in the proof of Proposition 3.1, we identify $(T_e^{0,1} SL(2, \mathbb{C}))^*$ with $(T_e^{1,0} SL(2, \mathbb{C}))^*$ by sending any u to \bar{u} .

As in the proof of Proposition 3.2, fixing a nonzero element of $\bigwedge^6(sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$, we get an $SL(2, \mathbb{C})$ -equivariant isomorphism of $\bigwedge^5(sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})^*$ with $sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. Using this isomorphism, we have

$$(3.5) \quad (d(\omega_h \wedge \omega_h))(e) \in sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}.$$

As noted in the proof of Proposition 3.2, the Kähler form ω_h is preserved by the left-translation action of $SU(2)$ on $SL(2, \mathbb{C})$. Consequently, the 5-form $d(\omega_h^2)$ is preserved by the left-translation action of $SU(2)$ on $SL(2, \mathbb{C})$. This implies that the element $(d(\omega_h^2))(e)$ in (3.5) is fixed by the adjoint action of $SU(2)$ on $sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. From this it follows that $(d(\omega_h^2))(e) = 0$, because $(sl(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})^{SU(2)} = sl(2, \mathbb{C})^{SU(2)} \otimes_{\mathbb{R}} \mathbb{C} = 0$. Since $d\omega_h^2$ is invariant under the right-translation action of $SL(2, \mathbb{C})$ on itself, and $(d(\omega_h^2))(e) = 0$, we conclude that $d(\omega_h^2) = 0$. \square

As before, $TSL(2, \mathbb{C})$ is the holomorphic tangent bundle of $TSL(2, \mathbb{C})$. Let ∇^h denote the Chern connection on $TSL(2, \mathbb{C})$ corresponding to the Hermitian structure h on $SL(2, \mathbb{C})$. The torsion of the connection ∇^h of $TSL(2, \mathbb{C})$ will be denoted by $\mathcal{T}(\nabla^h)$; it is a C^∞ section of $\Omega_{SL(2, \mathbb{C})}^{2,0} \otimes (TSL(2, \mathbb{C}))$.

Consider the Hermitian structure h on $TSL(2, \mathbb{C})$. It produces a C^∞ isomorphism

$$h' : \Omega_{SL(2, \mathbb{C})}^{1,0} \longrightarrow TSL(2, \mathbb{C})$$

defined by $h(h'(w), v) = w(v)$ for $w \in (\Omega_{SL(2, \mathbb{C})}^{1,0})_x$, $v \in T_x SL(2, \mathbb{C})$ and $x \in SL(2, \mathbb{C})$. We note that h' is a conjugate linear isomorphism. Using the isomorphism h' , the torsion $\mathcal{T}(\nabla^h)$ is a C^∞ section of $(\bigwedge^2(TSL(2, \mathbb{C}))) \otimes (TSL(2, \mathbb{C}))$.

Proposition 3.4. *The torsion $\mathcal{T}(\nabla^h) \in C^\infty(SL(2, \mathbb{C}), (\bigwedge^2(TSL(2, \mathbb{C}))) \otimes (TSL(2, \mathbb{C})))$ lies in the subspace*

$$C^\infty(SL(2, \mathbb{C}), \bigwedge^3(TSL(2, \mathbb{C}))) \subset C^\infty(SL(2, \mathbb{C}), (\bigwedge^2(TSL(2, \mathbb{C}))) \otimes (TSL(2, \mathbb{C}))).$$

In other words, the torsion is totally skew-symmetric.

The holonomy of the connection ∇^h lies in $SU(3)$.

Proof. Consider the element

$$(3.6) \quad \mathcal{T}(\nabla^h)(e) \in (\bigwedge^2 sl(2, \mathbb{C})) \otimes sl(2, \mathbb{C}),$$

where $e \in SL(2, \mathbb{C})$ is the identity element. It is invariant under the adjoint action of $SU(2)$ because the Hermitian structure h is preserved by the left translation action of $SU(2)$ on $SL(2, \mathbb{C})$.

Let V_0 be the standard two dimensional representation of $SU(2)$. The $SU(2)$ -module $sl(2, \mathbb{C})$ is isomorphic to the symmetric product $\text{Sym}^2(V_0)$.

Therefore, the $SU(2)$ -module in (3.6) is isomorphic to $(\bigwedge^2 \text{Sym}^2(V_0)) \otimes \text{Sym}^2(V_0)$. But

$$\bigwedge^2 \text{Sym}^2(V_0) = \text{Sym}^2(V_0)$$

(see [FH, p. 160, Ex. 11.35]), and

$$\text{Sym}^2(V_0) \otimes \text{Sym}^2(V_0) = \text{Sym}^4(V_0) \oplus \text{Sym}^2(V_0) \oplus \text{Sym}^0(V_0)$$

(see [FH, p. 151, Ex. 11.11]). Consequently,

$$((\bigwedge^2 \text{Sym}^2(V_0)) \otimes \text{Sym}^2(V_0))^{\text{SU}(2)} = \text{Sym}^0(V_0) = \bigwedge^3 \text{Sym}^2(V_0).$$

Consequently, $\mathcal{T}(\nabla^h)$ is a section of $\bigwedge^3(T\text{SL}(2, \mathbb{C}))$. This proves the first part of the proposition.

To prove the second part of the proposition, consider the Hermitian structure on the trivial holomorphic line bundle $\bigwedge^3(T\text{SL}(2, \mathbb{C}))$ induced by h . It is a constant Hermitian structure on the trivial holomorphic line bundle. Hence the holonomy of the connection on $\bigwedge^3(T\text{SL}(2, \mathbb{C}))$ induced by ∇^h is trivial. Consequently, the holonomy of the connection ∇^h lies in the subgroup $SU(3) \subset U(3)$. \square

4. A CLASS OF SOLUTIONS OF THE STROMINGER SYSTEM

Let

$$(4.1) \quad \Gamma \subset \text{SL}(2, \mathbb{C})$$

be a cocompact lattice, meaning Γ is a closed discrete subgroup of $\text{SL}(2, \mathbb{C})$ such that the quotient

$$(4.2) \quad M := \text{SL}(2, \mathbb{C})/\Gamma$$

is compact. We note that M is *not* a Kähler manifold.

Since the Hermitian structure h on $\text{SL}(2, \mathbb{C})$ constructed in Section 3 is invariant under the right-translation action of $\text{SL}(2, \mathbb{C})$ on itself, we conclude that h defines a Hermitian structure on M . Let \widehat{h} denote the Hermitian structure on M given by h . Note that the pullback of \widehat{h} by the quotient map $\text{SL}(2, \mathbb{C}) \rightarrow M$ coincides with h . Let

$$(4.3) \quad \omega \in C^\infty(M, \Omega_M^{1,1})$$

be the Kähler form on M associated to \widehat{h} . Let

$$(4.4) \quad \nabla^\omega$$

be the Chern connection on TM associated to ω .

Corollary 4.1. *The differential form ω in (4.3) satisfies the identity*

$$d(\omega^2) = 0.$$

Proof. Since the pullback of ω to $\text{SL}(2, \mathbb{C})$, by the quotient map $\text{SL}(2, \mathbb{C}) \rightarrow M$, coincides with ω_h , from Lemma 3.3 it follows that $d(\omega^2) = 0$. \square

For any torsionfree coherent analytic sheaf F on M , let $\det(F)$ be the determinant line bundle on M ; see [Ko, Ch. V, § 6] for the construction of the determinant bundle. Define the *degree* of F to be

$$(4.5) \quad \text{degree}(F) := \int_M \alpha(F) \wedge \omega \wedge \omega \in \mathbb{R},$$

where $\alpha(F)$ is any 2-form on M representing the first Chern class $c_1(\det(F)) \in H^2(M, \mathbb{R})$.

Lemma 4.2. *The degree is well defined.*

Proof. Let α and β be two 2-forms on M representing $c_1(\det(F))$. So, $\alpha - \beta = d\delta$, where δ is a smooth 1-form on M . Now,

$$\int_M \alpha \wedge \omega^2 - \int_M \beta \wedge \omega^2 = \int_M (\alpha - \beta) \wedge \omega^2 = \int_M (d\delta) \wedge \omega^2 = \int_M \delta \wedge d(\omega^2) = 0$$

be Corollary 4.1. So, $\int_M \alpha \wedge \omega^2 = \int_M \beta \wedge \omega^2$. Hence the degree is independent of the choice of the differential form representing the first Chern class. \square

Since the connection ∇^ω is the descent of the connection ∇^h considered in Proposition 3.4, the following corollary is an immediate consequence of Proposition 3.4.

Corollary 4.3. *The torsion of the connection ∇^ω is a C^∞ section of $\bigwedge^3 TM$; in other words, the torsion is totally skew-symmetric.*

The holonomy of the connection ∇^ω lies in $\mathrm{SU}(3)$.

We note that the torsion of the connection ∇^ω is nonzero because M is not Kähler.

We choose Γ such that there are irreducible unitary representations of Γ .

Remark 4.4. There are many examples of such Γ ; see [La, p. 3393, Theorem 2.1]. Note that any free nonabelian group has irreducible unitary representations in $\mathrm{U}(n)$ for all $n \geq 2$. To see this, take any two elements g_1 and g_2 of $\mathrm{SU}(n)$ such that $g_1 g_2 g_1^{-1} g_2^{-1}$ is a generator of the center of $\mathrm{SU}(n)$. The subgroup of $\mathrm{U}(n)$ generated by g_1 and g_2 is irreducible.

Let

$$(4.6) \quad \rho : \Gamma \longrightarrow \mathrm{U}(n)$$

be an irreducible representation; this means that the only linear subspaces of \mathbb{C}^n left invariant by the action of $\rho(\Gamma)$ are 0 and \mathbb{C}^n . Let

$$(4.7) \quad (E, \nabla) \longrightarrow M$$

be the unitary flat vector bundle over M given by ρ . We briefly recall the constructions of the vector bundle E and the connection ∇ on it. Consider the trivial vector bundle $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^n$ on $\mathrm{SL}(2, \mathbb{C})$; it has the trivial connection. This trivial connection is unitary with respect to the standard inner product on \mathbb{C}^n . The group Γ acts on $\mathrm{SL}(2, \mathbb{C})$ as right-translations, and it acts on \mathbb{C}^n as follows: the action of any $\gamma \in \Gamma$ sends any $v \in \mathbb{C}^n$ to

$\rho(\gamma^{-1})(v)$. Consider the diagonal action of Γ on $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^n$ constructed using these two actions. Let $(\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^n)/\Gamma$ be the quotient for this action. The natural map

$$(\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^n)/\Gamma \longrightarrow \mathrm{SL}(2, \mathbb{C})/\Gamma = M$$

is a vector bundle, which we will denote by E . The trivial connection on the vector bundle $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^n \longrightarrow \mathrm{SL}(2, \mathbb{C})$ descends to a flat unitary connection on E ; this descended connection on E will be denoted by ∇ .

A holomorphic vector bundle F of positive rank on M is called *stable* if for every nonzero coherent analytic subsheaf $V \subset F$ with $\mathrm{rank}(V) < \mathrm{rank}(F)$, the inequality

$$\frac{\mathrm{degree}(V)}{\mathrm{rank}(V)} < \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)}$$

holds, where degree is defined in (4.5) (and Lemma 4.2).

Proposition 4.5. *The holomorphic vector bundle E over M in (4.7) is stable.*

Proof. Since the vector bundle E admits a flat connection (recall that ∇ is flat), we have $c_1(\det(E)) = c_1(E) = 0$. Hence $\mathrm{degree}(E) = 0$.

Since the connection ∇ in (4.7) is unitary flat and irreducible, the proof of Proposition 8.2 in [Ko, page 176] gives that E is stable. In fact, the proof of Proposition 8.2 in [Ko, page 176], which is for irreducible Einstein-Hermitian bundles, gets simplified due to the stronger input that ∇ is unitary flat. \square

Let $\{A_0, B_0, C_0\}$ be the basis of $\mathfrak{sl}(2, \mathbb{C})$ defined by

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $A_0 \wedge B_0 \wedge C_0$ is a nonzero element of the line $\bigwedge^3 \mathfrak{sl}(2, \mathbb{C})$; we will call this element θ_0 . Note that the adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on $\bigwedge^3 \mathfrak{sl}(2, \mathbb{C})$ preserves θ_0 , because the action of $\mathrm{SL}(2, \mathbb{C})$ on $\bigwedge^3 \mathfrak{sl}(2, \mathbb{C})$ is trivial (the group $\mathrm{SL}(2, \mathbb{C})$ does not have any nontrivial character).

The holomorphic tangent bundle $T\mathrm{SL}(2, \mathbb{C})$ of $\mathrm{SL}(2, \mathbb{C})$ is identified with the trivial vector bundle $\mathrm{SL}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ using right-translation invariant vector fields. This identification produces a holomorphic isomorphism of the holomorphic tangent bundle TM , where M is constructed in (4.2), with the trivial vector bundle $M \times \mathfrak{sl}(2, \mathbb{C})$. Using this isomorphism, the above element $\theta_0 \in \bigwedge^3 \mathfrak{sl}(2, \mathbb{C})$ produces a trivialization of the canonical line bundle

$$K_M := \bigwedge^3 \Omega_M^{3,0} = (\bigwedge^3 TM)^*.$$

Let

$$(4.8) \quad \theta \in H^0(M, K_M)$$

be the nowhere zero holomorphic section given by θ_0 .

Theorem 4.6. *Consider the sextuple $(M, \theta, \omega, \nabla^\omega, E, \nabla)$ constructed in (4.2), (4.8), (4.3), (4.4) and (4.7). It solves the Strominger system. Moreover, it solves the equation of motion.*

Proof. Since ∇ is flat, the equations in (2.1) are satisfied.

The differential forms on both sides of equation (2.2) are given by right-translation invariant 1-forms on $SL(2, \mathbb{C})$. Moreover, these two 1-forms on $SL(2, \mathbb{C})$ are invariant under the left-translation action of $SU(2)$. Hence both sides of equation (2.2) vanish identically by Proposition 3.1.

The two form $\|\Omega\|_\omega \cdot \omega^2$ is given by a right-translation invariant 1-forms on $SL(2, \mathbb{C})$ which is also fixed by the left-translation action of $SU(2)$ on $SL(2, \mathbb{C})$. Therefore, by Proposition 3.2, the form $\|\Omega\|_\omega \cdot \omega^2$ is a constant scalar multiple of ω^2 . Hence $d(\|\Omega\|_\omega \cdot \omega^2) = 0$ by Corollary 4.1.

The two 2-forms on two sides of equation (2.4) are given by right-translation invariant 2-forms on $SL(2, \mathbb{C})$ that are fixed by the left-translation action of $SU(2)$ on $SL(2, \mathbb{C})$. Therefore, from Proposition 3.2 we conclude that (2.4) holds.

Therefore, the sextuple $(M, \theta, \omega, \nabla^\omega, E, \nabla)$ solves the Strominger system. We will now show that equation (2.5) also holds.

Let $R(\nabla^\omega)$ be the curvature of the connection ∇^ω on TM . Since ∇^ω is the Chern connection for ω , we have

$$R(\nabla^\omega)^{2,0} = 0 = R(\nabla^\omega)^{0,2}.$$

To prove that $R(\nabla^\omega) \wedge \omega^2 = 0$, we first note that $R(\nabla^\omega) \wedge \omega^2 = 0$ if and only if

$$\star_\omega(R(\nabla^\omega) \wedge \omega^2) = 0,$$

where \star_ω is the star operator on differential forms on M constructed using ω ; we note that $\star_\omega(R(\nabla^\omega) \wedge \omega^2)$ is a C^∞ section of $\text{End}(TM) = TM \otimes (TM)^*$. Using the identification of

Consider the evaluation $\star_\omega(R(\nabla^\omega) \wedge \omega^2)(e) \in \text{End}(T_e M)$ of $\star_\omega(R(\nabla^\omega) \wedge \omega^2)$ at $e \in SL(2, \mathbb{C})$. Using the identification of $T_e M$ with $sl(2, \mathbb{C})$, it will be considered as an element of

$$\text{End}(sl(2, \mathbb{C})) = sl(2, \mathbb{C}) \otimes sl(2, \mathbb{C})^*.$$

The space of invariants $\text{End}(sl(2, \mathbb{C}))^{SU(2)} \subset \text{End}(sl(2, \mathbb{C}))$ is one dimensional, and it is generated by the identity element $\text{Id}_{sl(2, \mathbb{C})}$. In other words, $\star_\omega(R(\nabla^\omega) \wedge \omega^2)(e)$ is a scalar multiple of $\text{Id}_{sl(2, \mathbb{C})}$. Let $\lambda \in \mathbb{C}$ be such that

$$(4.9) \quad \star_\omega(R(\nabla^\omega) \wedge \omega^2)(e) = \lambda \cdot \text{Id}_{sl(2, \mathbb{C})}.$$

Since $\star_\omega(R(\nabla^\omega) \wedge \omega^2)$ is given by a section of $\text{End}(TSL(2, \mathbb{C}))$ which is invariant under the right-translation action of $SL(2, \mathbb{C})$ on itself, from (4.9) we conclude that

$$(4.10) \quad \star_\omega(R(\nabla^\omega) \wedge \omega^2)(e) = \lambda \cdot \text{Id}_{TM}.$$

From (4.10) it follows immediately that

$$(4.11) \quad R(\nabla^\omega) \bigwedge \omega^2 = \lambda \cdot \text{Id}_{TM} \otimes \omega^3.$$

Since $\star_\omega(R(\nabla^\omega) \bigwedge \omega^2)$ is given by a section of $\text{End}(T\text{SL}(2, \mathbb{C}))$ which is invariant under the right-translation action of $\text{SL}(2, \mathbb{C})$ on itself, to prove that $\star_\omega(R(\nabla^\omega) \bigwedge \omega^2) = 0$, it suffices to show that $\lambda = 0$, where λ is the scalar in (4.9).

To prove that $\lambda = 0$, first that $c_1(TM) = 0$, because TM is holomorphically trivial. Hence

$$\text{trace}(R(\nabla^\omega)) = d\beta$$

for some smooth 1-form β on M . Therefore,

$$(4.12) \quad \int_M \text{trace}(R(\nabla^\omega)) \wedge \omega^2 = \int_M (d\beta) \wedge \omega^2 = \int_M \beta \wedge d(\omega^2) = 0$$

by Lemma 3.3. Now from (4.11),

$$\int_M \text{trace}(R(\nabla^\omega)) \wedge \omega^2 = 3\lambda \cdot \int_M \omega^3.$$

Since $\int_M \omega^3 \neq 0$, from (4.12) we conclude that $\lambda = 0$. Therefore, (2.5) holds. This completes the proof. \square

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